Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 314 (2008) 558–570
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Using a new discretization approach to design a delayed LQG controller

M. Haraguchi*, H.Y. Hu

MOE Key Lab of Structure Mechanics and Control for Aircraft, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, People's Republic of China

Received 17 January 2007; received in revised form 10 October 2007; accepted 14 January 2008
Handling Editor: J. Lam
Available online 7 March 2008

Abstract

In general, discrete-time controls have become more and more preferable in engineering because of their easy implementation and simple computations. However, the available discretization approaches for the systems having time delays increase the system dimensions and have a high computational cost. This paper presents an effective discretization approach for the continuous-time systems with an input delay. The approach enables one to transform the input-delay system into a delay-free system, but retain the system dimensions unchanged in the state transformation. To demonstrate an application of the approach, this paper presents the design of an LQ regulator for continuous-time systems with an input delay and gives a state observer with a Kalman filter for estimating the full-state vector from some measurements of the system as well. The case studies in the paper well support the efficacy and efficiency of the proposed approach applied to the vibration control of a three-story structure model with the actuator delay taken into account.

1. Introduction

Digital controllers, widely used in control practice, have shown a great number of advantages, such as their accuracy and universality, over analog controllers in control performance. One of the limits to the performance and applications of digital controllers is the unavoidable time delays due to the computation of control strategies and related digital filters [1,2]. Over the past few decades, the effect of time delays on system dynamics has drawn considerable attention in various fields [3–5]. The time-delay systems are described by delay differential equations, which have some unique features different from ordinary differential equations. For example, no matter how short the time delays are, the time-delay systems have an infinite number of characteristic roots. Such infiniteness usually makes the analysis and the synthesis of the time-delay systems very difficult.

This paper focuses on the time delays existing in the system input. The presence of the input delays, if not considered in a controller design, may lead to deterioration of the control performance or may even cause

*Corresponding author. Fax: +86 25 84891512.
E-mail address: masakazu_haraguchi@nifty.com (M. Haraguchi).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2008.01.047

instability of the system. This is because the control force may act at the exact moment when the controlled system does not need it. The frequently used ways of dealing with the input-delay systems are to convert them into delay-free ordinary systems by using state transformations. For example, Kwon and Pearson [6] proposed a state transformation in a continuous-time framework, and obtained a stabilizing controller. Their extensions to various systems, such as time-varying systems and uncertain systems, have been available [7–9]. These controls overcome some problems of the conventional Smith predictor method [5], and can stabilize the unstable systems. However, the integrations involved in the state transformations lead to complicated computations, which are impractical for their implementation. Choi and Chung [10], and Kim and his coworkers [11] obtained stabilizing controllers of another type. Their controllers, the so-called memoryless controllers, are designed to guarantee the delay-independent stability of the closed-loop systems, and just have the feedback of the current state only. Hence, the memoryless controllers are very easy to implement. Nevertheless, the memoryless controllers may be unduly conservative and have less control performance as compared with the controllers using the information of the time delays and employing the feedback of the past control history as well as the current state [8]. Cai and Huang [12,13], and Zhou et al. [14] proposed the approaches converting the input-delay problems into delay-free problems in discrete-time frameworks, and designing the linear quadratic (LQ) regulator. Their discrete-time approaches need only simple computations during the control processes, and may suppress the vibrations of the systems well. However, the dimensions of a controlled system after their state transformations are much higher than those of the original time-delay system, especially when the system involves multiple inputs or when the time delay is long. Thus, it is highly preferable to find a more tractable control approach to the systems having input delays.

The primary aim of this paper is to present an effective state transformation, based on Ref. [15], and its application to the problem of a linear quadratic Gaussian (LQG) control for a continuous-time system of multiple degrees of freedom with an input delay. The proposed approach does not increase the system dimensions with the state transformation, and involves simple computations during the control process. The rest of the paper is organized as follows. In Section 2, the state transformation is presented first. Then, the design problems of an LQ regulator and a state observer for the system after the transformation are discussed in Section 3. In Section 4, some case studies are given for the vibration control of a three-story structure with the actuator delay considered to show the efficacy of the proposed approach. Finally, the concluding remarks are made in Section 5.

2. State transformation for an input-delay system

The system of concern is a continuous-time system of multiple degrees of freedom with an input delay described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{B}_c\mathbf{u}(t-\tau) + \mathbf{v}_c(t), \tag{1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{w}_c(t), \tag{2}$$

where $\tau > 0$ is the time delay, and $\mathbf{x}(t) \in \mathscr{R}^{\ell\times 1}$, $\mathbf{u}(t) \in \mathscr{R}^{m\times 1}$, $\mathbf{y}(t) \in \mathscr{R}^{n\times 1}$ are the state vector, the input vector and the output vector, respectively. $\mathbf{v}_c(t)$ and $\mathbf{w}_c(t)$ are the stochastic processes having the mean values of zero and the incremental covariances $\mathbf{R}_{1c}\,\mathrm{d}t$ and $\mathbf{R}_{2c}\,\mathrm{d}t$, respectively. The system can be converted, as in Appendix A, into the following discrete-time form with the sampling period Δ:

$$\mathbf{x}(k+1) = \mathbf{A}_d\mathbf{x}(k) + \mathbf{B}_{d1}\mathbf{u}(k-\alpha+1) + \mathbf{B}_{d2}\mathbf{u}(k-\alpha) + \mathbf{v}_d(k), \tag{3}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{w}_d(k). \tag{4}$$

In what follows, this discrete-time form of the input-delay system is converted into a delay-free system by introducing a state transformation. The transformation is given in two subsections for $0 < \tau \leqslant \Delta$ and $\Delta < \tau$, respectively.

2.1. Case 1: $0<\tau\leqslant\Delta$

When the time delay τ is shorter than or equal to the sampling period Δ (i.e., $\alpha=1$), the state vector at the $(k+\alpha)$th sampling can be written as

$$\begin{aligned}\mathbf{x}(k+\alpha) &= \mathbf{A}_d\mathbf{x}(k)+\mathbf{B}_{d2}\mathbf{u}(k-1)+\mathbf{B}_{d1}\mathbf{u}(k)+\mathbf{v}_d(k)\\ &= \overline{\mathbf{x}}(k)+\mathbf{B}_{d1}\mathbf{u}(k)+\mathbf{v}_d(k),\end{aligned}\tag{5}$$

where the vector $\overline{\mathbf{x}}(k)$ is defined as

$$\overline{\mathbf{x}}(k)\equiv\mathbf{A}_d\mathbf{x}(k)+\mathbf{B}_{d2}\mathbf{u}(k-1).\tag{6}$$

$\overline{\mathbf{x}}(k+1)$ can be obtained as follows by substituting Eq. (3) into Eq. (6):

$$\begin{aligned}\overline{\mathbf{x}}(k+1) &= \mathbf{A}_d\mathbf{x}(k+1)+\mathbf{B}_{d2}\mathbf{u}(k)\\ &= \mathbf{A}_d\{\mathbf{A}_d\mathbf{x}(k)+\mathbf{B}_{d2}\mathbf{u}(k-1)\}+(\mathbf{A}_d\mathbf{B}_{d1}+\mathbf{B}_{d2})\mathbf{u}(k)+\mathbf{A}_d\mathbf{v}_d(k)\\ &= \mathbf{A}_d\overline{\mathbf{x}}(k)+\overline{\mathbf{B}}_d\mathbf{u}(k)+\mathbf{A}_d\mathbf{v}_d(k),\end{aligned}\tag{7}$$

where $\overline{\mathbf{B}}_d$ is

$$\overline{\mathbf{B}}_d\equiv\mathbf{A}_d\mathbf{B}_{d1}+\mathbf{B}_{d2}.\tag{8}$$

Eq. (7) is in the standard form of a linear system in control theory and does not have any time delay apparently. In other words, a delay-free system in the state space can be obtained from input-delay system (1) when $0<\tau\leqslant\Delta$ by introducing a new state vector.

2.2. Case 2: $\Delta<\tau$

When the time delay τ is longer than the sampling period Δ (i.e., $\alpha\geqslant 2$), Eq. (1) can be converted as follows [15]. From Eq. (3), the state vector $\mathbf{x}$ at the $(k+\alpha)$th sampling can be obtained as

$$\begin{aligned}\mathbf{x}(k+\alpha) &= \mathbf{A}_d\mathbf{x}(k+\alpha-1)+\mathbf{B}_{d1}\mathbf{u}(k)+\mathbf{B}_{d2}\mathbf{u}(k-1)+\mathbf{v}_d(k+\alpha-1)\\ &= \mathbf{A}_d\{\mathbf{A}_d\mathbf{x}(k+\alpha-2)+\mathbf{B}_{d1}\mathbf{u}(k-1)+\mathbf{B}_{d2}\mathbf{u}(k-2)+\mathbf{v}_d(k+\alpha-2)\}\\ &\quad+\mathbf{B}_{d1}\mathbf{u}(k)+\mathbf{B}_{d2}\mathbf{u}(k-1)+\mathbf{v}_d(k+\alpha-1)\\ &= \cdots\\ &= \mathbf{A}_d^{\alpha}\mathbf{x}(k)+\mathbf{A}_d[\mathbf{I},\mathbf{A}_d,\ldots,\mathbf{A}_d^{\alpha-2}]\begin{bmatrix}\mathbf{B}_{d1}\mathbf{u}(k-1)\\ \mathbf{B}_{d1}\mathbf{u}(k-2)\\ \vdots\\ \mathbf{B}_{d1}\mathbf{u}(k-\alpha+1)\end{bmatrix}+\mathbf{B}_{d1}\mathbf{u}(k)\\ &\quad+[\mathbf{I},\mathbf{A}_d,\ldots,\mathbf{A}_d^{\alpha-1}]\begin{bmatrix}\mathbf{B}_{d2}\mathbf{u}(k-1)\\ \mathbf{B}_{d2}\mathbf{u}(k-2)\\ \vdots\\ \mathbf{B}_{d2}\mathbf{u}(k-\alpha)\end{bmatrix}+[\mathbf{I},\mathbf{A}_d,\ldots,\mathbf{A}_d^{\alpha-1}]\begin{bmatrix}\mathbf{v}_d(k+\alpha-1)\\ \mathbf{v}_d(k+\alpha-2)\\ \vdots\\ \mathbf{v}_d(k)\end{bmatrix}\\ &= \overline{\mathbf{x}}(k)+\mathbf{B}_{d1}\mathbf{u}(k)+[\mathbf{I},\mathbf{A}_d,\ldots,\mathbf{A}_d^{\alpha-1}]\begin{bmatrix}\mathbf{v}_d(k+\alpha-1)\\ \mathbf{v}_d(k+\alpha-2)\\ \vdots\\ \mathbf{v}_d(k)\end{bmatrix},\end{aligned}\tag{9}$$

where the vector $\bar{\mathbf{x}}(k)$ is defined as

$$\bar{\mathbf{x}}(k) \equiv \mathbf{A}_d^{\alpha}\mathbf{x}(k) + \mathbf{A}_d[\mathbf{I}, \mathbf{A}_d, \ldots, \mathbf{A}_d^{\alpha-2}]\begin{bmatrix} \mathbf{B}_{d1}\mathbf{u}(k-1) \\ \mathbf{B}_{d1}\mathbf{u}(k-2) \\ \vdots \\ \mathbf{B}_{d1}\mathbf{u}(k-\alpha+1) \end{bmatrix} + [\mathbf{I}, \mathbf{A}_d, \ldots, \mathbf{A}_d^{\alpha-1}]\begin{bmatrix} \mathbf{B}_{d2}\mathbf{u}(k-1) \\ \mathbf{B}_{d2}\mathbf{u}(k-2) \\ \vdots \\ \mathbf{B}_{d2}\mathbf{u}(k-\alpha) \end{bmatrix}. \tag{10}$$

Substituting Eq. (3) into Eq. (10) yields

$$\begin{aligned}\bar{\mathbf{x}}(k+1) = {} & \mathbf{A}_d^{\alpha}\{\mathbf{A}_d\mathbf{x}(k) + \mathbf{B}_{d1}\mathbf{u}(k-\alpha+1) + \mathbf{B}_{d2}\mathbf{u}(k-\alpha) + \mathbf{v}_d(k)\} \\ & + \mathbf{A}_d[\mathbf{I}, \mathbf{A}_d, \ldots, \mathbf{A}_d^{\alpha-2}]\begin{bmatrix} \mathbf{B}_{d1}\mathbf{u}(k) \\ \mathbf{B}_{d1}\mathbf{u}(k-1) \\ \vdots \\ \mathbf{B}_{d1}\mathbf{u}(k-\alpha+2) \end{bmatrix} + [\mathbf{I}, \mathbf{A}_d, \ldots, \mathbf{A}_d^{\alpha-1}]\begin{bmatrix} \mathbf{B}_{d2}\mathbf{u}(k) \\ \mathbf{B}_{d2}\mathbf{u}(k-1) \\ \vdots \\ \mathbf{B}_{d2}\mathbf{u}(k-\alpha+1) \end{bmatrix}.\end{aligned} \tag{11}$$

By recasting Eq. (11) in terms of $\mathbf{u}(k)$, one obtains

$$\begin{aligned}\bar{\mathbf{x}}(k+1) &= \mathbf{A}_d\left\{\mathbf{A}_d^{\alpha}\mathbf{x}(k) + \mathbf{A}_d[\mathbf{I}, \mathbf{A}_d, \ldots, \mathbf{A}_d^{\alpha-2}]\begin{bmatrix} \mathbf{B}_{d1}\mathbf{u}(k-1) \\ \mathbf{B}_{d1}\mathbf{u}(k-2) \\ \vdots \\ \mathbf{B}_{d1}\mathbf{u}(k-\alpha+1) \end{bmatrix} + [\mathbf{I}, \mathbf{A}_d, \ldots, \mathbf{A}_d^{\alpha-1}]\begin{bmatrix} \mathbf{B}_{d2}\mathbf{u}(k-1) \\ \mathbf{B}_{d2}\mathbf{u}(k-2) \\ \vdots \\ \mathbf{B}_{d2}\mathbf{u}(k-\alpha) \end{bmatrix}\right\} + \mathbf{A}_d\mathbf{B}_{d1}\mathbf{u}(k) + \mathbf{B}_{d2}\mathbf{u}(k) + \mathbf{A}_d^{\alpha}\mathbf{v}_d(k) \\ &= \mathbf{A}_d\bar{\mathbf{x}}(k) + (\mathbf{A}_d\mathbf{B}_{d1} + \mathbf{B}_{d2})\mathbf{u}(k) + \mathbf{A}_d^{\alpha}\mathbf{v}_d(k).\end{aligned} \tag{12}$$

With the help of $\overline{\mathbf{B}}_d \equiv \mathbf{A}_d\mathbf{B}_{d1} + \mathbf{B}_{d2}$, one obtains

$$\bar{\mathbf{x}}(k+1) = \mathbf{A}_d\bar{\mathbf{x}}(k) + \overline{\mathbf{B}}_d\mathbf{u}(k) + \mathbf{A}_d^{\alpha}\mathbf{v}_d(k). \tag{13}$$

As can be seen from Eq. (13), input-delay system (3) can also be converted into a delay-free system when $\alpha \geqslant 2$, by introducing the new state vector $\bar{\mathbf{x}}(k)$ defined in Eq. (10).

2.3. Remarks

The new state vectors in Eqs. (6) and (10) have the dimensions of

$$\dim\{\bar{\mathbf{x}}(k)\} = \ell \times 1. \tag{14}$$

This implies that the dimensions of the delay-free system in the state transformation remain the same as those of the original system. In contrast, the approaches in Refs. [12–14,16] have to increase the dimensions of the delay-free system to $\ell + m\alpha$ because of their state transformations, and hence, give rise to very high computational costs, especially when the original system involves multiple inputs or when the time delay is relatively long.

3. LQG controller for a system with an input delay

3.1. LQ regulator

This subsection focuses on the design of the LQ regulator for the input-delay system discussed in Section 2 since the LQ regulator, as a classical controller, represents a convenient approach to optimal and robust control for engineering systems. The cost function for the LQ regulator reads

$$J = \sum_{k=0}^{+\infty} \{\mathbf{x}^{\mathrm{T}}(k)\mathbf{Q}_1\mathbf{x}(k) + \mathbf{u}^{\mathrm{T}}(k)\mathbf{Q}_2\mathbf{u}(k)\} = J_1 + J_2, \tag{15}$$

where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are the weighting matrices which are symmetric and positive semidefinite, and J_1 and J_2 are as follows:

$$J_1 \equiv \sum_{k=0}^{+\infty} \mathbf{x}^{\mathrm{T}}(k)\mathbf{Q}_1\mathbf{x}(k), \quad J_2 \equiv \sum_{k=0}^{+\infty} \mathbf{u}^{\mathrm{T}}(k)\mathbf{Q}_2\mathbf{u}(k). \tag{16}$$

If the target system does not have any time delay, the control input of the LQ regulator should be $\mathbf{u}(k) = -\mathbf{L}(k)\mathbf{x}(k)$. In this case, the substitution of the input vector into the cost function leads to a quadratic form with respect to the state vector $\mathbf{x}(k)$. For the present case, however, the LQ regulator is designed for state-space models (7) and (13) in the new state vector $\overline{\mathbf{x}}(k)$. Thus, the control input of the LQ regulator becomes $\mathbf{u}(k) = -\mathbf{L}(k)\overline{\mathbf{x}}(k)$. Hence, cost function (15) is not unified in the quadratic form of the original state vector $\mathbf{x}(k)$ or the form of the new state vector $\overline{\mathbf{x}}(k)$. As a result, cost function (15) cannot be used directly for the design of the LQ regulator for systems (7) and (13). In what follows, the cost function in a quadratic form of $\overline{\mathbf{x}}(k)$ is derived from cost function (15). For this purpose, J_1 can be divided into the following two parts:

$$\begin{aligned} J_1 &= \sum_{k=0}^{+\infty} \mathbf{x}^{\mathrm{T}}(k)\mathbf{Q}_1\mathbf{x}(k) \\ &= \sum_{k=0}^{\alpha-1} \mathbf{x}^{\mathrm{T}}(k)\mathbf{Q}_1\mathbf{x}(k) + \sum_{k=\alpha}^{+\infty} \mathbf{x}^{\mathrm{T}}(k)\mathbf{Q}_1\mathbf{x}(k) = J_{11} + J_{12}, \end{aligned} \tag{17}$$

where

$$J_{11} \equiv \sum_{k=0}^{\alpha-1} \mathbf{x}^{\mathrm{T}}(k)\mathbf{Q}_1\mathbf{x}(k), \quad J_{12} \equiv \sum_{k=\alpha}^{+\infty} \mathbf{x}^{\mathrm{T}}(k)\mathbf{Q}_1\mathbf{x}(k). \tag{18}$$

The study begins with J_{11}. In the design of the LQ regulator, $\mathbf{v}_d(k)$ in Eq. (3) can be neglected. Thus, one can simply consider

$$\mathbf{x}(k+1) = \mathbf{A}_d\mathbf{x}(k) + \mathbf{B}_{d1}\mathbf{u}(k-\alpha+1) + \mathbf{B}_{d2}\mathbf{u}(k-\alpha). \tag{19}$$

Provided $\mathbf{u}(k) = 0$ when $k<0$, Eq. (19) becomes

$$\mathbf{x}(k+1) = \mathbf{A}_d\mathbf{x}(k) = \mathbf{A}_d^2\mathbf{x}(k-1) = \cdots = \mathbf{A}_d^{k+1}\mathbf{x}(0) \tag{20}$$

in the summation range of J_{11} because the terms of $\mathbf{u}(k-\alpha+1)$ and $\mathbf{u}(k-\alpha)$ disappear. Substituting Eq. (20) into J_{11} in Eq. (18) yields

$$J_{11} = \sum_{k=0}^{\alpha-1} \mathbf{x}^{\mathrm{T}}(k)\mathbf{Q}_1\mathbf{x}(k) = \sum_{k=0}^{\alpha-1} \mathbf{x}^{\mathrm{T}}(0)(\mathbf{A}_d^k)^{\mathrm{T}}\mathbf{Q}_1\mathbf{A}_d^k\mathbf{x}(0). \tag{21}$$

Eq. (21) shows that J_{11} is a constant determined by the initial state $\mathbf{x}(0)$, and independent of the input vector $\mathbf{u}(k)$. The summation in J_{12} starts from $k = \alpha$ to $+\infty$, and hence J_{12} can be written as

$$J_{12} = \sum_{k=\alpha}^{+\infty} \mathbf{x}^{\mathrm{T}}(k)\mathbf{Q}_1\mathbf{x}(k) = \sum_{i=0}^{+\infty} \mathbf{x}^{\mathrm{T}}(i+\alpha)\mathbf{Q}_1\mathbf{x}(i+\alpha). \tag{22}$$

If $\mathbf{v}_d(k)$ is neglected, Eqs. (5) and (9) yield

$$\mathbf{x}(k+\alpha) = \overline{\mathbf{x}}(k) + \mathbf{B}_{d1}\mathbf{u}(k). \tag{23}$$

Substituting Eq. (23) into Eq. (22) leads to

$$\begin{aligned} J_{12} &= \sum_{i=0}^{+\infty} \{\overline{\mathbf{x}}(i) + \mathbf{B}_{d1}\mathbf{u}(i)\}^{\mathrm{T}} \mathbf{Q}_1 \{\overline{\mathbf{x}}(i) + \mathbf{B}_{d1}\mathbf{u}(i)\} \\ &= \sum_{i=0}^{+\infty} \{\overline{\mathbf{x}}^{\mathrm{T}}(i)\mathbf{Q}_1\overline{\mathbf{x}}(i) + 2\overline{\mathbf{x}}^{\mathrm{T}}(i)\mathbf{Q}_1\mathbf{B}_{d1}\mathbf{u}(i) + \mathbf{u}^{\mathrm{T}}(i)\mathbf{B}_{d1}^{\mathrm{T}}\mathbf{Q}_1\mathbf{B}_{d1}\mathbf{u}(i)\}. \end{aligned} \tag{24}$$

Now, cost function (15) can be recast as

$$\begin{aligned} J &= J_{11} + J_{12} + J_2 \\ &= J_{11} + \sum_{k=0}^{+\infty} \{\overline{\mathbf{x}}^{\mathrm{T}}(k)\mathbf{Q}_1\overline{\mathbf{x}}(k) + 2\overline{\mathbf{x}}^{\mathrm{T}}(k)\mathbf{Q}_1\mathbf{B}_{d1}\mathbf{u}(k) + \mathbf{u}^{\mathrm{T}}(k)(\mathbf{B}_{d1}^{\mathrm{T}}\mathbf{Q}_1\mathbf{B}_{d1} + \mathbf{Q}_2)\mathbf{u}(k)\} \\ &= J_{11} + \overline{J}, \end{aligned} \tag{25}$$

where

$$\overline{J} \equiv \sum_{k=0}^{+\infty} \{\overline{\mathbf{x}}^{\mathrm{T}}(k)\mathbf{Q}_1\overline{\mathbf{x}}(k) + 2\overline{\mathbf{x}}^{\mathrm{T}}(k)\overline{\mathbf{Q}}_{12}\mathbf{u}(k) + \mathbf{u}^{\mathrm{T}}(k)\overline{\mathbf{Q}}_2\mathbf{u}(k)\}, \tag{26}$$

$$\overline{\mathbf{Q}}_{12} \equiv \mathbf{Q}_1\mathbf{B}_{d1}, \quad \overline{\mathbf{Q}}_2 \equiv \mathbf{B}_{d1}^{\mathrm{T}}\mathbf{Q}_1\mathbf{B}_{d1} + \mathbf{Q}_2. \tag{27}$$

Because $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are the symmetric and positive semidefinite matrices, $\overline{\mathbf{Q}}_2$ also becomes symmetric and positive semidefinite. The optimization problem of minimizing the cost function J becomes the corresponding problem of $\overline{J}$ since J_{11} is a constant. In other words, the control input minimizing $\overline{J}$ also minimizes the cost function J. Moreover, the cost function $\overline{J}$ becomes a quadratic form of the state $\overline{\mathbf{x}}(k)$ if the control input is given by $\mathbf{u}(k) = -\mathbf{L}(k)\overline{\mathbf{x}}(k)$. Thus, the optimization problem of minimizing $\overline{J}$ can be dealt with as an LQ regulator problem with respect to the state vector $\overline{\mathbf{x}}(k)$. The control input minimizing the cost function $\overline{J}$ is

$$\begin{aligned} \mathbf{u}(k) &= -\mathbf{L}(k)\overline{\mathbf{x}}(k) \\ &= -\{\overline{\mathbf{B}}_d^{\mathrm{T}}\mathbf{S}(k+1)\overline{\mathbf{B}}_d + \overline{\mathbf{Q}}_2\}^{-1}\{\overline{\mathbf{B}}_d^{\mathrm{T}}\mathbf{S}(k+1)\mathbf{A}_d + \overline{\mathbf{Q}}_{12}^{\mathrm{T}}\}\overline{\mathbf{x}}(k), \end{aligned} \tag{28}$$

where $\mathbf{S}(k)$ is the solution of the discrete-time Riccati equation

$$\begin{aligned} \mathbf{S}(k) = \mathbf{A}_d^{\mathrm{T}}\mathbf{S}(k+1)\mathbf{A}_d - \{\mathbf{A}_d^{\mathrm{T}}\mathbf{S}(k+1)\overline{\mathbf{B}}_d + \overline{\mathbf{Q}}_{12}\} \\ \times \{\overline{\mathbf{B}}_d^{\mathrm{T}}\mathbf{S}(k+1)\overline{\mathbf{B}}_d + \overline{\mathbf{Q}}_2\}^{-1}\{\overline{\mathbf{B}}_d^{\mathrm{T}}\mathbf{S}(k+1)\mathbf{A}_d + \overline{\mathbf{Q}}_{12}^{\mathrm{T}}\} + \mathbf{Q}_1. \end{aligned} \tag{29}$$

This yields a time-varying controller. The stationary controller can be obtained as

$$\mathbf{u}(k) = -\mathbf{L}\overline{\mathbf{x}}(k) = -(\overline{\mathbf{B}}_d^{\mathrm{T}}\mathbf{S}\overline{\mathbf{B}}_d + \overline{\mathbf{Q}}_2)^{-1}(\overline{\mathbf{B}}_d^{\mathrm{T}}\mathbf{S}\mathbf{A}_d + \overline{\mathbf{Q}}_{12}^{\mathrm{T}})\overline{\mathbf{x}}(k), \tag{30}$$

where $\mathbf{S}$ is determined by the following Riccati equation:

$$\mathbf{S} = \mathbf{A}_d^{\mathrm{T}}\mathbf{S}\mathbf{A}_d - (\mathbf{A}_d^{\mathrm{T}}\mathbf{S}\overline{\mathbf{B}}_d + \overline{\mathbf{Q}}_{12})(\overline{\mathbf{B}}_d^{\mathrm{T}}\mathbf{S}\overline{\mathbf{B}}_d + \overline{\mathbf{Q}}_2)^{-1}(\overline{\mathbf{B}}_d^{\mathrm{T}}\mathbf{S}\mathbf{A}_d + \overline{\mathbf{Q}}_{12}^{\mathrm{T}}) + \mathbf{Q}_1. \tag{31}$$

In order to design LQ regulator (30), Riccati equation (31) needs to be solved. In general, it is impossible to solve the Riccati equation analytically. However, Matlab command (`dare`) can offer numerical results and complete the design of the stationary controller (30).

3.2. State observer with a Kalman filter

The LQ regulator requires a measurable full-state vector [17], which may not be available in most practical cases. This subsection discusses the observer with the Kalman filter for estimating the state vector $\overline{\mathbf{x}}(k)$ from the available measurements of the system described by Eqs. (1) and (2). All of the vectors and the coefficient matrices on the right-hand sides of Eqs. (6) and (10) defining the state vector $\overline{\mathbf{x}}(k)$ can be obtained at the kth sampling except for $\mathbf{x}(k)$. Thus, if the unknown vector $\mathbf{x}(k)$ can be estimated, the state vector $\overline{\mathbf{x}}(k)$ can be obtained. Now one can postulate the observer as follows to estimate the state vector $\mathbf{x}(k)$:

$$\hat{\mathbf{x}}(k+1|k) = \mathbf{A}_d\hat{\mathbf{x}}(k|k-1) + \mathbf{B}_{d1}\mathbf{u}(k-\alpha+1) + \mathbf{B}_{d2}\mathbf{u}(k-\alpha) + \mathbf{K}(k)\{\mathbf{y}(k) - \mathbf{C}\hat{\mathbf{x}}(k|k-1)\}, \tag{32}$$

where $\hat{\mathbf{x}}(k+1|k)$ represents the prediction of $\mathbf{x}(k+1)$ estimated by using the kth measurement. If one defines the estimation error as $\mathbf{e} \equiv \mathbf{x} - \hat{\mathbf{x}}$, the estimation error at the $(k+1)$st sampling reads

$$\begin{aligned}\mathbf{e}(k+1) &= \mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1|k)\\ &= \mathbf{A}_d\{\mathbf{x}(k) - \hat{\mathbf{x}}(k|k-1)\} + \mathbf{v}_d(k) - \mathbf{K}(k)[\mathbf{C}\{\mathbf{x}(k) - \hat{\mathbf{x}}(k|k-1)\} - \mathbf{w}_d(k)]\\ &= [\mathbf{I},\ \ -\mathbf{K}(k)]\begin{bmatrix}\mathbf{A}_d\mathbf{e}(k) + \mathbf{v}_d(k)\\ \mathbf{C}\mathbf{e}(k) + \mathbf{w}_d(k)\end{bmatrix}.\end{aligned} \tag{33}$$

The observer can minimize the following variance of the estimation error:

$$\mathbf{P}(k) \equiv \mathrm{E}\{\mathbf{e}(k)\mathbf{e}^{\mathrm{T}}(k)\}. \tag{34}$$

From Eqs. (33) and (34), $\mathbf{P}(k+1)$ becomes

$$\begin{aligned}\mathbf{P}(k+1) &= \mathrm{E}\{\mathbf{e}(k+1)\mathbf{e}^{\mathrm{T}}(k+1)\}\\ &= \mathrm{E}\left\{[\mathbf{I},\ \ -\mathbf{K}(k)]\begin{bmatrix}\mathbf{A}_d\mathbf{e}(k) + \mathbf{v}_d(k)\\ \mathbf{C}\mathbf{e}(k) + \mathbf{w}_d(k)\end{bmatrix}\begin{bmatrix}\mathbf{A}_d\mathbf{e}(k) + \mathbf{v}_d(k)\\ \mathbf{C}\mathbf{e}(k) + \mathbf{w}_d(k)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{I}\\ -\mathbf{K}^{\mathrm{T}}(k)\end{bmatrix}\right\}.\end{aligned} \tag{35}$$

Because $\mathbf{e}(k)$ is independent of $\mathbf{v}_d(k)$ and $\mathbf{w}_d(k)$, Eq. (35) can be recast in the following form by using Eq. (34):

$$\mathbf{P}(k+1) = [\mathbf{I},\ \ -\mathbf{K}(k)]\begin{bmatrix}\mathbf{A}_d\mathbf{P}(k)\mathbf{A}_d^{\mathrm{T}} + \mathbf{R}_1 & \mathbf{A}_d\mathbf{P}(k)\mathbf{C}^{\mathrm{T}} + \mathbf{R}_{12}\\ \mathbf{C}\mathbf{P}(k)\mathbf{A}_d^{\mathrm{T}} + \mathbf{R}_{12}^{\mathrm{T}} & \mathbf{C}\mathbf{P}(k)\mathbf{C}^{\mathrm{T}} + \mathbf{R}_2\end{bmatrix}\begin{bmatrix}\mathbf{I}\\ -\mathbf{K}^{\mathrm{T}}(k)\end{bmatrix}. \tag{36}$$

According to the idea of the completion of squares [17], Eq. (36) can be minimized if $\mathbf{K}(k)$ satisfies

$$\mathbf{K}(k)\{\mathbf{C}\mathbf{P}(k)\mathbf{C}^{\mathrm{T}} + \mathbf{R}_2\} = \mathbf{A}_d\mathbf{P}(k)\mathbf{C}^{\mathrm{T}} + \mathbf{R}_{12}. \tag{37}$$

If $\{\mathbf{C}\mathbf{P}(k)\mathbf{C}^{\mathrm{T}} + \mathbf{R}_2\}$ is positive definite, one obtains the Kalman gain as follows:

$$\mathbf{K}(k) = \{\mathbf{A}_d\mathbf{P}(k)\mathbf{C}^{\mathrm{T}} + \mathbf{R}_{12}\}\{\mathbf{C}\mathbf{P}(k)\mathbf{C}^{\mathrm{T}} + \mathbf{R}_2\}^{-1}. \tag{38}$$

Substituting Eq. (38) into Eq. (36) results in the following discrete-time Riccati equation:

$$\mathbf{P}(k+1) = \mathbf{A}_d\mathbf{P}(k)\mathbf{A}_d^{\mathrm{T}} + \mathbf{R}_1 - \{\mathbf{A}_d\mathbf{P}(k)\mathbf{C}^{\mathrm{T}} + \mathbf{R}_{12}\}\{\mathbf{C}\mathbf{P}(k)\mathbf{C}^{\mathrm{T}} + \mathbf{R}_2\}^{-1}\{\mathbf{C}\mathbf{P}(k)\mathbf{A}_d^{\mathrm{T}} + \mathbf{R}_{12}^{\mathrm{T}}\}. \tag{39}$$

By solving Eq. (39), the Kalman gain in Eq. (38) can be determined. The corresponding steady-state solution satisfies the following algebraic Riccati equation:

$$\mathbf{P} = \mathbf{A}_d\mathbf{P}\mathbf{A}_d^{\mathrm{T}} + \mathbf{R}_1 - (\mathbf{A}_d\mathbf{P}\mathbf{C}^{\mathrm{T}} + \mathbf{R}_{12})(\mathbf{C}\mathbf{P}\mathbf{C}^{\mathrm{T}} + \mathbf{R}_2)^{-1}(\mathbf{C}\mathbf{P}\mathbf{A}_d^{\mathrm{T}} + \mathbf{R}_{12}^{\mathrm{T}}). \tag{40}$$

The solution of Eq. (40) gives the steady-state Kalman gain as follows:

$$\mathbf{K} = (\mathbf{A}_d\mathbf{P}\mathbf{C}^{\mathrm{T}} + \mathbf{R}_{12})(\mathbf{C}\mathbf{P}\mathbf{C}^{\mathrm{T}} + \mathbf{R}_2)^{-1}. \tag{41}$$

3.3. Summary of controller design for an input-delay system

The LQG control for the systems having an input delay can be realized in the following way. The control input is determined by LQ regulator (30), where the regulator gain can be obtained by solving Riccati equation (31), and the new state vector $\overline{\mathbf{x}}(k)$ is defined in Eqs. (6) and (10) for $\alpha = 1$ and $\alpha \geqslant 2$, respectively. Because such an LQ regulator includes the linear summations only, the computation during the control process is very simple. In addition, the vector $\mathbf{x}(k)$ in the state vector $\overline{\mathbf{x}}(k)$ is estimated by Kalman filter (32) if the full-state vector cannot be measured.

The design parameters of the regulator are the weighting matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ corresponding to the original state vector $\mathbf{x}(k)$ and the input vector $\mathbf{u}(k)$, respectively. Hence, one can choose them in a similar way to design the regulator and need not consider the physical meanings of the new state vector $\overline{\mathbf{x}}(k)$.

4. Illustrative examples

To demonstrate the efficacy and efficiency of the proposed approach, this section presents some numerical simulations on the vibration control of a three-story building [12] with a delayed actuator, as shown in Fig. 1. In the numerical simulations, the vibration of the structure was suppressed by the proposed control approach. The acceleration data of the El Centro earthquake (north-south component) scaled to the maximum acceleration of $0.12g\,\mathrm{m\,s^{-2}}$ shown in Fig. 2 were used as the external excitation. The mass, stiffness and damping coefficients of each story unit were taken as $m_i = 1000\,\mathrm{kg}$, $k_i = 980\,\mathrm{kN\,m^{-1}}$ and $c_i = 1.407\,\mathrm{kNs\,m^{-1}}$, respectively ($i = 1, \ldots, 3$). An actuator was assumed to be installed on the first-story unit to apply the active control force u. The control systems for suppressing the vibrations of huge structures usually use hydraulic actuators because of their strong output. However, such type of actuators generally have remarkable time delays between the output force and the input signal [2]. Thus, it was assumed that the actuator in the model has a time delay τ. A sensor was placed at the third-story unit to measure the inter-story drift x_3. The drift x_3 is the only one measurable output of the system. The vectors $\mathbf{x}(t)$, $\mathbf{v}_c(t)$, the matrices $\mathbf{A}_c$, $\mathbf{B}_c$ and $\mathbf{C}$ in Eqs. (1) and (2) read

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{X}(t) \\ \dot{\mathbf{X}}(t) \end{bmatrix}, \quad \mathbf{v}_c(t) = \begin{bmatrix} \mathbf{0} \\ \mathbf{m}^{-1}\mathbf{h}_p \end{bmatrix} p(t),$$
$$\mathbf{A}_c = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{m}^{-1}\mathbf{k} & -\mathbf{m}^{-1}\mathbf{c} \end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix} \mathbf{0} \\ \mathbf{m}^{-1}\mathbf{h}_u \end{bmatrix},$$
$$\mathbf{C} = [0,\ 0,\ 1,\ 0,\ 0,\ 0], \tag{42}$$

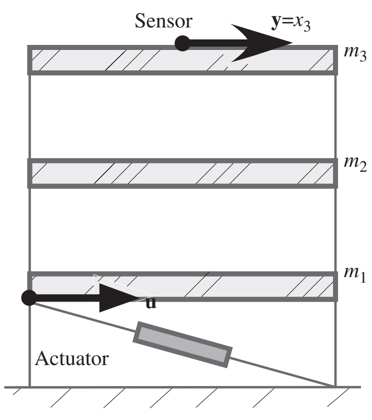

Fig. 1. Three-story building model with a delayed actuator.

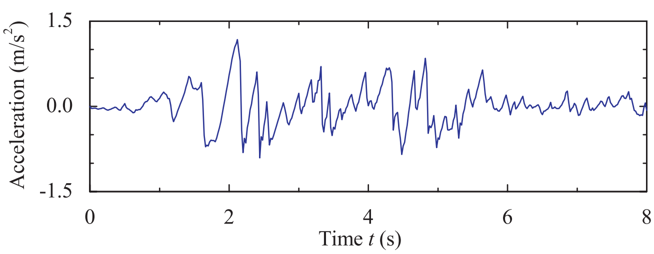

Fig. 2. El Centro earthquake acceleration for external excitation.

where $\mathbf{X}(t) = [x_1(t),\ x_2(t),\ x_3(t)]^T$, $p(t)$ is the external excitation and $\mathbf{m}$, $\mathbf{k}$, $\mathbf{c}$, $\mathbf{h}_u$ and $\mathbf{h}_p$ are:

$$\mathbf{m} = 10^3 \times \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{k} = 9.8 \cdot 10^5 \times \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\mathbf{c} = 1407 \times \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{h}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{h}_p = -10^3 \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{43}$$

The weighting matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ for the design of the LQ regulator were chosen to be $\mathbf{Q}_1 = \mathrm{diag}([10^5, 10^5, 10^5, 1, 1, 1])$ and $\mathbf{Q}_2 = 2 \times 10^{-10}$, respectively. The weighting matrices $\mathbf{R}_1$, $\mathbf{R}_{12}$ and $\mathbf{R}_2$ for the observer design were fixed as $\mathbf{R}_1 = \mathrm{diag}([0, 0, 0, 2 \times 10^{-1}, 0, 0])$, $\mathbf{R}_{12} = \mathbf{0}$ and $\mathbf{R}_2 = 10^{-6}$, respectively. The sampling period was taken as $\Delta = 0.002\,\mathrm{s}$.

Fig. 3 gives the variations of the maximum inter-story drifts of each story unit and the control input with an increase of the time delay τ when the normal LQG control system was applied, in which the time delay was not taken into account in the controller design. The dashed lines represent the case without control, and the solid lines represent the case when the normal LQG controller was applied. As can be observed from Fig. 3, the normal LQG controller makes the system unstable even if the time delay τ is very short.

Fig. 4 shows the time histories of the inter-story drifts of each unit and the control input when the proposed control strategy was applied to the case when the time delay $\tau = 0.1\,\mathrm{s}$. The dashed lines denote the case without control, and the solid lines for the case with the proposed control strategy. Fig. 4 shows that the vibration of each story unit can be well suppressed even though the control system can sense only the drift of the third story unit (Fig. 4c). This result indicates that the Kalman estimator works well for the control system having a time delay, and the proposed control strategy is effective for such a system even if the full-state vector is not measurable.

Fig. 5 gives the variations of the maximum inter-story drifts of the each story unit and the control input with an increase of the time delay τ when the proposed control strategy was applied. The control efficiency

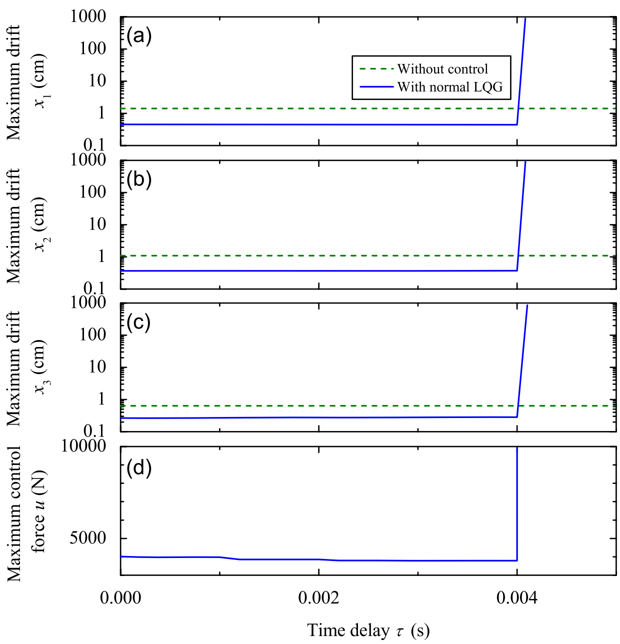

Fig. 3. Variation of maximum inter-story drifts (a)–(c) and control force (d) with an increase of time delay. - - -: without control, and —: under normal LQG without the delay taken into account.

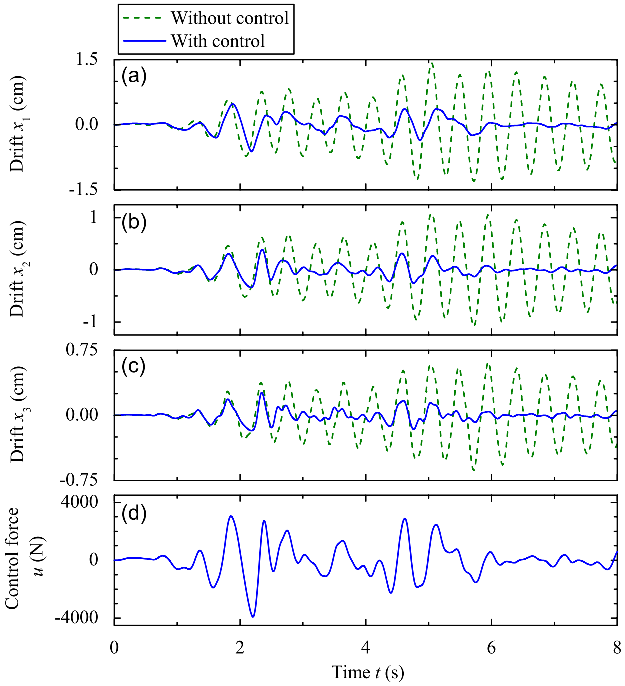

Fig. 4. Time histories of inter-story drifts (a)–(c) and control force (d). - - -: without control, and —: proposed control ($\tau = 0.1$ s).

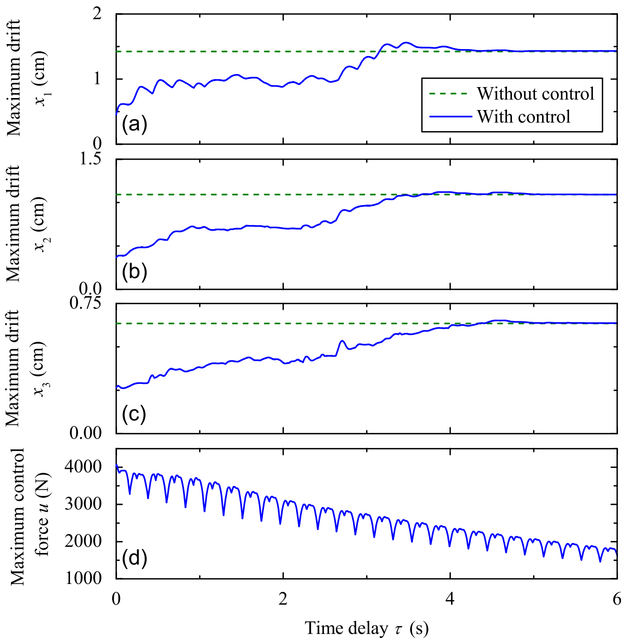

Fig. 5. Variation of maximum inter-story drifts (a)–(c) and control force (d) with an increase of time delay. - - -: without control, and —: proposed control.

decreases gradually with an increase of the time delay τ, and the maximum drifts converge with those of the open-loop response. The figure shows that the system does not cause instability over the whole time delay range ($0 \leqslant \tau \leqslant 6$ s, that is, $0 \leqslant \alpha \leqslant 3000$).

In order to make comparisons, other simulations based on two existing methods were performed. Firstly, the proposed control strategy was compared with the discrete-time control presented by Cai et al. [12] under the same condition. Cai's controller can be used only if the full-state vector is available since it is an LQ regulator. In this comparison, hence, it was assumed that the full-state vector can be measured. The simulation result shows that the proposed control strategy has the same control efficiency as Cai's control upon the assumption. However, Cai's control has to increase the system dimensions through the state transformation. For example, when $\tau = 0.1\,\mathrm{s}$, Cai's control and the other existing discrete-time controls in [14,17] increase the system dimensions to $\ell + m\alpha = 56$. In contrast, the proposed approach retains the same dimensions $\ell = 6$ of the delay-free system as the dimensions of the original system. That is, the proposed approach can achieve the same control effect as Cai's control, but does not increase the system dimensions.

The second comparison focuses on Choi's memoryless controller [10]. The memoryless controllers, designed to satisfy the delay-dependent stability conditions, just need the feedback of the current state. Hence, they are easier to be used for practical implementation than other controls including the proposed control strategy if they are applicable to the system of concern. Choi's control needs iterations for finding the positive-definite solution of the Riccati-like equation. According to his algorithm, the iteration starts from determining whether there exists a solution of the equation with one set of the weighting matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ and the parameter $\varepsilon > 0$. If the solution cannot be found, the procedure is iterated with the new ε replaced by $\varepsilon/2$ until the positive-definite solution is found or ε is less than the prescribed lower limit. For this simulation, the previously mentioned parameters were used, and another specific parameter γ in this control was set to be $\gamma = 0.5$. Although the iteration was done from $\varepsilon = 1$ to 10^{-20}, the solution could not be found. Hence, the controller could not be designed for this system. Choi pointed out in his paper that even though the algorithm fails for one choice of the weighting matrices, it cannot be concluded that it fails for their other choices. Nevertheless, the changes of the weighting matrices indicate the change of the prescribed specification of the controller, which is inadequate for controller designs.

5. Conclusions

The paper presents an applicable approach to the design of an LQG controller for multi-degree-of-freedom systems with a time delay in control input. The approach enables one to transform the input-delay system into a delay-free system first and then complete the design of an LQG controller for the input-delay system as is usually done for a delay-free system. The proposed approach features the state transform, which retains the system dimensions unchanged, whereas the existing approaches have to increase the system dimensions, especially when there are multiple inputs or when the input delay is long. The proposed approach is applicable to systems with a long time delay because the controller is established without any approximations to the time delay. As shown in the design of the LQ regulator and the state observer, the proposed approach is simple and easy to implement. Furthermore, the numerical simulations indicate that the proposed approach can effectively control the vibration of systems with a time delay even when the full-state vector of the system is not measurable.

Acknowledgments

This research was supported in part by the National Natural Science Foundation of China under the Grant of 10532050. The first author wishes to thank Professor Zaihua Wang of Nanjing University of Aeronautics and Astronautics for his helpful comments on the manuscript.

Appendix A. Derivation of discrete-time form

In order to obtain the discrete-time form of Eqs. (1) and (2), one applies the zero-order holder with the sampling period Δ [12,16,17]. The time delay τ can be written with Δ as

$$\tau = \alpha\Delta - \beta, \tag{A.1}$$

where $\alpha > 0$ is an integer and β satisfies $0 \leqslant \beta < \Delta$. The analytical solution of Eq. (1) is

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}_c(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \mathrm{e}^{\mathbf{A}_c(t-\xi)}\mathbf{B}_c\mathbf{u}(\xi - \tau)\,\mathrm{d}\xi + \int_{t_0}^{t} \mathrm{e}^{\mathbf{A}_c(t-\xi)}\mathbf{v}_c(\xi)\,\mathrm{d}\xi. \tag{A.2}$$

Here, assuming $t_0 = k\Delta$ and $t = (k+1)\Delta$, and substituting them into Eq. (A.2), one obtains

$$\mathbf{x}\{(k+1)\Delta\} = \mathrm{e}^{\mathbf{A}_c\Delta}\mathbf{x}(k\Delta) + \int_{k\Delta}^{(k+1)\Delta} \mathrm{e}^{\mathbf{A}_c\{(k+1)\Delta-\xi\}}\mathbf{B}_c\mathbf{u}(\xi - \tau)\,\mathrm{d}\xi + \int_{k\Delta}^{(k+1)\Delta} \mathrm{e}^{\mathbf{A}_c\{(k+1)\Delta-\xi\}}\mathbf{v}_c(\xi)\,\mathrm{d}\xi. \tag{A.3}$$

Replacing the integration variable ξ by $\eta = (k+1)\Delta - \xi$ leads to

$$\mathbf{x}\{(k+1)\Delta\} = \mathrm{e}^{\mathbf{A}_c\Delta}\mathbf{x}(k\Delta) + \int_0^{\Delta} \mathrm{e}^{\mathbf{A}_c\eta}\mathbf{B}_c\mathbf{u}(k\Delta + \Delta - \tau - \eta)\,\mathrm{d}\eta + \int_0^{\Delta} \mathrm{e}^{\mathbf{A}_c\eta}\mathbf{v}_c(k\Delta + \Delta - \eta)\,\mathrm{d}\eta. \tag{A.4}$$

Substituting Eq. (A.1) into Eq. (A.4) yields

$$\mathbf{x}\{(k+1)\Delta\} = \mathrm{e}^{\mathbf{A}_c\Delta}\mathbf{x}(k\Delta) + \int_0^{\Delta} \mathrm{e}^{\mathbf{A}_c\eta}\mathbf{B}_c\mathbf{u}\{(k-\alpha)\Delta + \Delta + \beta - \eta\}\,\mathrm{d}\eta + \int_0^{\Delta} \mathrm{e}^{\mathbf{A}_c\eta}\mathbf{v}_c(k\Delta + \Delta - \eta)\,\mathrm{d}\eta. \tag{A.5}$$

Noticing that the vectors are discretized as $\mathbf{x}(t) = \mathbf{x}(k\Delta)$ when $k\Delta \leqslant t < (k+1)\Delta$ by the zero-order holder, one can know that the term of $\mathbf{u}$ in Eq. (A.5) can be divided into two cases

$$\mathbf{u}\{(k-\alpha)\Delta + \Delta + \beta - \eta\} = \begin{cases} \mathbf{u}\{(k-\alpha+1)\Delta\} & \text{if } 0 \leqslant \eta \leqslant \beta, \\ \mathbf{u}\{(k-\alpha)\Delta\} & \text{if } \beta < \eta \leqslant \Delta. \end{cases} \tag{A.6}$$

According to Eq. (A.6), Eq. (A.5) can be written as

$$\begin{aligned} \mathbf{x}\{(k+1)\Delta\} = {} & \mathrm{e}^{\mathbf{A}_c\Delta}\mathbf{x}(k\Delta) + \int_0^{\beta} \mathrm{e}^{\mathbf{A}_c\eta}\,\mathrm{d}\eta\,\mathbf{B}_c\mathbf{u}\{(k-\alpha+1)\Delta\} \\ & + \int_{\beta}^{\Delta} \mathrm{e}^{\mathbf{A}_c\eta}\,\mathrm{d}\eta\,\mathbf{B}_c\mathbf{u}\{(k-\alpha)\Delta\} + \int_0^{\Delta} \mathrm{e}^{\mathbf{A}_c\eta}\mathbf{v}_c(k\Delta + \Delta - \eta)\,\mathrm{d}\eta. \end{aligned} \tag{A.7}$$

Denoting $\mathbf{x}(k\Delta)$ by $\mathbf{x}(k)$, the discrete-time forms can be obtained as Eqs. (3) and (4), where

$$\begin{cases} \mathbf{A}_d = \mathrm{e}^{\mathbf{A}_c\Delta}, \\ \mathbf{B}_{d1} = \int_0^{\beta} \mathrm{e}^{\mathbf{A}_c\eta}\,\mathrm{d}\eta\,\mathbf{B}_c, \\ \mathbf{B}_{d2} = \int_{\beta}^{\Delta} \mathrm{e}^{\mathbf{A}_c\eta}\,\mathrm{d}\eta\,\mathbf{B}_c, \\ \mathbf{v}_d(k) = \int_0^{\Delta} \mathrm{e}^{\mathbf{A}_c\eta}\mathbf{v}_c(k\Delta + \Delta - \eta)\,\mathrm{d}\eta, \end{cases} \tag{A.8}$$

whereas $\mathbf{v}_d$ and $\mathbf{w}_d$ are the Gaussian white noises with the zero-mean values and the covariances as

$$\begin{cases} \mathrm{E}\{\mathbf{v}_d(k)\mathbf{v}_d^{\mathrm{T}}(k)\} = \int_0^{\Delta} \mathrm{e}^{\mathbf{A}_c\eta}\mathbf{R}_{1c}\mathrm{e}^{\mathbf{A}_c^{\mathrm{T}}\eta}\,\mathrm{d}\eta = \mathbf{R}_1, \\ \mathrm{E}\{\mathbf{v}_d(k)\mathbf{w}_d^{\mathrm{T}}(k)\} = \mathbf{R}_{12}, \\ \mathrm{E}\{\mathbf{w}_d(k)\mathbf{w}_d^{\mathrm{T}}(k)\} = \mathbf{R}_2. \end{cases} \tag{A.9}$$

References

[1] H.Y. Hu, Z.H. Wang, Stability analysis of damped SDOF systems with two time delays in state feedback, *Journal of Sound and Vibration* 214 (2) (1998) 213–225.

[2] H.Y. Hu, Z.H. Wang, *Dynamics of Controlled Mechanical Systems with Delayed Feedback*, Springer, Berlin, 2002.

[3] G. Stépán, *Retarded Dynamical Systems: Stability and Characteristic Functions*, Longman Scientific and Technical, 1989.

[4] L. Dugard, E.I. Verriest, *Stability and Control of Time-Delay Systems*, Springer, Berlin, 1998.

[5] S.I. Niculescu, *Delay Effects on Stability: A Robust Control Approach*, Springer, Berlin, 2001.

[6] W.H. Kwon, A.E. Pearson, Feedback stabilization of linear systems with delayed control, *IEEE Transactions on Automatic Control* AC-25 (2) (1980) 266–269.
[7] Z. Artstein, Linear systems with delayed controls: a reduction, *IEEE Transaction on Automatic Control* AC-27 (4) (1982) 869–879.
[8] Y.S. Moon, P.G. Park, W.H. Kwon, Robust stabilization of uncertain input-delayed systems using reduction method, *Automatica* 37 (2001) 307–312.
[9] D. Popescu, V.l. Răsvan, Robustness properties of a control procedure for systems with input delay, *IEEE MELECON* (2004) 391–394.
[10] H.H. Choi, M.J. Chung, Memoryless H_∞ controller design for linear systems with delayed state and control, *Automatica* 31 (6) (1995) 917–919.
[11] J.H. Kim, E.T. Jeung, H.B. Park, Robust control for parameter uncertain delay systems in state and control input, *Automatica* 32 (9) (1996) 1337–1339.
[12] G. Cai, J. Huang, Optimal control method with time delay in control, *Journal of Sound and Vibration* 251 (3) (2002) 383–394.
[13] G.P. Cai, S.X. Yang, A discrete optimal control method for a flexible cantilever beam with time delay, *Journal of Vibration and Control* 12 (5) (2006) 509–526.
[14] H.Q. Zhou, L.S. Shieh, Q.G. Wang, C.R. Liu, State-space digital PID controller design for linear stochastic multivariable systems with input delay, *International Conference on Control and Automation* (2005) 840–845.
[15] K. Watanabe, *Control of Time-Delay Systems*, Corona Publishing, Japan, 1992 (in Japanese).
[16] Z.Q. Sun, *Theory and Application of Computer Control*, Tsinghua University Press, China, 1989 (in Chinese).
[17] K.J. Aström, B. Wittenmark, *Computer-Controlled Systems: Theory and Design*, third ed., Prentice-Hall, Englewood Cliffs, NJ, 1997.